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Renormalization and fixed points in Hilbert space

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Abstract

The energies of low-lying bound states of a microscopic quantum many-body system of particles can be worked out in a reduced Hilbert space. We present here and test a specific non-perturbative algorithm. We also show that real exceptional points which may be present in the spectrum can be identified as fixed points of coupling constants in the truncation procedure.

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1. Introduction

The construction of a rigorous microscopic quantum many-body theory able to describe bound particle systems such as molecules, atoms, aggregates, atomic nuclei and condensed systems has developed over a long period of time starting in the 1960s [1].

In practice, the explicit resolution of the problem necessitates the diagonalization of the many-body Hamiltonian in Hilbert space which is spanned by a complete set of basis states, in principle of infinite dimension, at least generally very large. In many cases the information of interest is, however, restricted to the knowledge of a few energetically low-lying states possessing collective properties which cannot be reproduced in the framework of quasi-particle descriptions. Hence, it would be convenient to work in a finite truncated subspace of the original Hilbert space. Many different procedures have been proposed and successfully applied, in particular in the study of structure properties of nuclei, atoms, molecules and condensed systems.

Rigorous projection methods lead to effective Hamiltonians which can in principle be explicitly generated by means of perturbation techniques [1–5, 16]. Unfortunately there exists no straightforward control on the convergence properties of the perturbation expansions which are involved, especially not when the interaction between the particles is strong as it is the case in atomic nuclei and strongly correlated systems such as quantum spin chains, ladders and networks for instance. Many attempts have been made in order to overcome this problem. Some authors developed cluster expansions, in particular Suzuki and Okamoto [7] and Kümmel *et al* [8]. Perturbation expansion resummations have been proposed [9, 10]. Pragmatic phenomenological procedures have also been introduced [11, 12]. More recently

effective 2-body interactions have been constructed in the framework of the nuclear many-body problem by means of a non-perturbative renormalization technique which cuts off the large momentum components of the interaction [13].

Perturbation expansions based on projection techniques [14, 15] diverge in specific but frequently encountered situations, in practice when the interaction between the constituents gets strong. This breakdown happens in the framework of the projection method when two states belonging to different subspaces, the subspace of the states which enter the description of the system and its complement to the total space, come arbitrarily close to or even cross each other [16].

The present investigations develop an approach which follows in spirit former work based on a renormalization concept [17–20]. We introduce a general and operational non-perturbative method for the treatment of the bound state many-body problem which leads to the generation of effective Hamiltonians in reduced spaces.

We first present the formal framework in which we develop the dimensional reduction procedure. We then define the method designed for many-body quantum systems at temperature $T = 0$ and test it on two models. Finally, we show the link which exists in this framework between exceptional points and fixed points appearing for some specific values of the interaction strength constants.

2. Formal framework

Consider a system with a fixed but arbitrary number of bound quantum objects (particles, spins) in a Hilbert space $\mathcal{H}^{(\mathcal{N})}$ of dimension N governed by a Hamiltonian $H^{(N)}\big(g_1^{(N)}, g_2^{(N)}, \ldots, g_p^{(N)}\big)$ where $\big\{g_1^{(N)}, g_2^{(N)}, \ldots, g_p^{(N)} \mapsto g^{(N)}\big\}$ are a set of parameters (coupling constants) which characterize $H^{(N)}$. The eigenvectors $\big|\Psi_i^{(N)}(g^{(N)})\big\rangle\{i = 1, \ldots, N\}$ of $H^{(N)}$ span the Hilbert space and are the solutions of the Schrödinger equation

$$H^{(N)}(g^{(N)})\big|\Psi_i^{(N)}(g^{(N)})\big\rangle = \lambda_i(g^{(N)})\big|\Psi_i^{(N)}(g^{(N)})\big\rangle. \tag{1}$$

The diagonalization of $H^{(N)}$ delivers both the eigenvalues $\{\lambda_i(g^{(N)}), i = 1, \ldots, N\}$ and eigenvectors $\big\{\big|\Psi_i^{(N)}(g^{(N)})\big\rangle, i = 1, \ldots, N\big\}$ in terms of a linear combination of orthogonal basis states $\{|\Phi_i\rangle, i = 1, \ldots, N\}$. Since $\dim \mathcal{H}^{(N)} = N$ is generally very large if not infinite and the information needed reduces to a finite part of the spectrum, it makes sense to try to restrict the space dimensions. If the relevant quantities of interest are for instance M eigenvalues out of the set $\{\lambda_i(g^{(N)})\}$, generally but not necessarily the lowest energy states, then one may define a new effective Hamiltonian $H^{(M)}(g^{(M)})$ such that

$$H^{(M)}(g^{(M)})\big|\Psi_i^{(M)}(g^{(M)})\big\rangle = \lambda_i(g^{(M)})\big|\Psi_i^{(M)}(g^{(M)})\big\rangle \tag{2}$$

with the constraints

$$\lambda_i(g^{(M)}) = \lambda_i(g^{(N)}) \tag{3}$$

for $i = 1, \ldots, M$. Equation (3) implies relations between the sets of coupling constants $g^{(M)}$ and $g^{(N)}$

$$g_k^{(M)} = f_k\big(g_1^{(N)}, g_2^{(N)}, \ldots, g_p^{(N)}\big) \tag{4}$$

with $k = 1, \ldots, p$. The solution of these equations generates new coupling constants which allow us to define a new Hamiltonian in the corresponding reduced space. Such an effective Hamiltonian $H^{(M)}(g^{(M)})$ may not be rigorously derivable from $H^{(N)}$ or unique. It should be constructed in such a way that it best preserves the eigenenergies and the properties of physical observables.

3. General space reduction procedure and renormalization algorithm for systems at temperature $T = 0$

We develop here an explicit and general approach which allows us to implement the former procedure. We start from the complete Hilbert space $\mathcal{H}^{(N)}$ in which the system is described by the Hamiltonian $H^{(N)}$. Since we want to reduce the dimensions of the space but describe the same physical system as in the original space, the Hamiltonian has to be changed and go over to an effective operator. This can in principle be achieved by means of the renormalization of quantities which characterize it, in practice for instance interaction strengths, coupling parameters. The evolution of these quantities with the reduction of space is determined by means of constraints which fix physical quantities such as energies or other physical observables corresponding to those obtained in the complete space and (or) experimentally known. In this way, the physical properties such as the energies of the low-energy part of the spectrum can be determined in the reduced space and are hopefully close to those which are generated in the complete space. In what follows, we consider a Hamiltonian with one parameter so that one needs one constraint to fix its value at each step of the reduction procedure. The quantity we consider to be fixed here is the ground state energy of the system. Constraints on the energies of other states can be implemented.

Following the procedure sketched above we reduce the dimensions of the space by means of a projection technique. Using the Feshbach formalism [14], we divide the Hilbert space $\mathcal{H}^{(N)}$ into two subspaces, $P\mathcal{H}^{(N)}$ and $Q\mathcal{H}^{(N)}$. In the present case the dimensions of the subspaces are chosen such that

$$\dim P\mathcal{H}^{(N)} = N - 1 \qquad \dim Q\mathcal{H}^{(N)} = 1. \tag{5}$$

In the projected subspace $P\mathcal{H}^{(N)}$ the system with energy E is described by the effective Hamiltonian [14, 15]

$$H_{\text{eff}}(E) = PHP + PHQ(E - QHQ)^{-1}QHP. \tag{6}$$

The Hamiltonian H characterizes the system in the Hilbert space $\mathcal{H}^{(N)}$. We suppose that H depends on one parameter g and write it in the form

$$H = H_0 + gH_1 \tag{7}$$

where H_0 and H_1 are Hamiltonian operators and g is a parameter (coupling constant) which generically characterizes the strength of the interaction between the constituents. It takes the value $g^{(N)}$ in $\mathcal{H}^{(N)}$. We consider an arbitrary complete set of basis states which spans $\mathcal{H}^{(N)}$ $\{|\Phi_i\rangle, i = 1, \ldots, N\}$. It may for instance be chosen as the eigenvectors of H_0 with the corresponding eigenvalues $\{\epsilon_i, i = 1, \ldots, N\}$.

The expression $H_{\text{eff}}(E)$ is generally the starting point of theories which rely on perturbation expansions [1]. Here we proceed differently. We consider

$$P\left|\Psi_1^{(N)}\right\rangle = \sum_{i=1}^{N-1} a_{1i}^{(N)}(g^{(N)})|\Phi_i\rangle \tag{8}$$

which is the projection on $P\mathcal{H}^{(N)}$ of an eigenvector

$$\left|\Psi_1^{(N)}\right\rangle = \sum_{i=1}^{N} a_{1i}^{(N)}(g^{(N)})|\Phi_i\rangle \tag{9}$$

of $\mathcal{H}^{(N)}$. If $\lambda_1^{(N)}$ is the eigenvalue corresponding to $\left|\Psi_1^{(N)}\right\rangle$ we look for the solution of

$$H_{\text{eff}}\left(\lambda_1^{(N)}\right)P\left|\Psi_1^{(N)}\right\rangle = \lambda_1^{(N)}P\left|\Psi_1^{(N)}\right\rangle. \tag{10}$$

We consider $P\left|\Psi_1^{(N)}\right\rangle$ to be the lowest energy eigenstate. In the one-dimensional subspace $Q\mathcal{H}^{(N)}$ any state can in principle be chosen, for instance a state which lies in the high energy sector of the spectrum of H_0. We impose the lowest eigenvalue in the $P\mathcal{H}^{(N)}$ subspace to be the same as the one in the complete space

$$\lambda_1^{(N-1)} = \lambda_1^{(N)}. \tag{11}$$

The expression of equation (10) is projected on $\langle\Phi_1|$ which is the eigenvector of H_0 with lowest energy

$$\langle\Phi_1|H_{\text{eff}}\left(\lambda_1^{(N)}\right)\left|P\Psi_1^{(N)}\right\rangle = \lambda_1^{(N)}(g^{(N)})a_{11}^{(N)}(g^{(N)}). \tag{12}$$

$H_{\text{eff}}\left(E = \lambda_1^{(N)}\right)$ is a non-local energy-dependent operator which contains the local operator H given by equation (7). In the expression, equation (6), of H_{eff} which appears in equation (12) we introduce $H^{(N-1)} = H_0 + gH_1$ with $g = g^{(N-1)}$ acting in the projected space $\mathcal{H}^{(N-1)}$. Equation (12) fixes the coupling constant $g^{(N-1)}$. Indeed, if one develops the left-hand member of the equation one gets

$$\langle\Phi_1|H_{\text{eff}}\left(\lambda_1^{(N)}\right)\left|P\Psi_1^{(N)}\right\rangle = \mathcal{F}(g^{(N-1)}) \tag{13}$$

where

$$\mathcal{F}(g^{(N-1)}) = \overline{H_{1N}^{(N-1)}} + H_{1N}^{(N-1)}\left(\lambda_1^{(N)} - H_{NN}^{(N-1)}\right)^{-1}\overline{H_{N1}^{(N-1)}} \tag{14}$$

with

$$H_{ij}^{(N-1)} = \langle\Phi_i|H^{(N-1)}|\Phi_j\rangle \tag{15}$$

and

$$\overline{H_{1N}^{(N-1)}} = \langle\Phi_1|H^{(N-1)}\left|P\Psi_1^{(N)}\right\rangle. \tag{16}$$

$\overline{H_{N1}^{(N-1)}}$ is the matrix element as $\overline{H_{1N}^{(N-1)}}$ with $\langle\Phi_1|$ replaced by $\langle\Phi_N|$.

Equation (12) can be worked out explicitly. The denominator in the second term of $H_{\text{eff}}\left(\lambda_1^{(N)}\right)$ of equation (14) is a scalar quantity since $\dim Q\mathcal{H}^{(N)} = 1$. Developing equation (13) leads to a relation which fixes a renormalized coupling constant $g^{(N-1)}$. One gets explicitly

$$a^{(N-1)}g^{(N-1)^2} + b^{(N-1)}g^{(N-1)} + c^{(N-1)} = 0 \tag{17}$$

where

$$a^{(N-1)} = G_{1N} - H_{NN}F_{1N} \tag{18}$$

with

$$H_{ij} = \langle\Phi_i|H_1|\Phi_j\rangle \tag{19}$$

$$b^{(N-1)} = a_{11}^{(N)}H_{NN}\left(\lambda_1^{(N)} - \epsilon_1\right) + F_{1N}\left(\lambda_1^{(N)} - \epsilon_N\right) \tag{20}$$

$$c^{(N-1)} = -a_{11}^{(N)}\left(\lambda_1^{(N)} - \epsilon_1\right)\left(\lambda_1^{(N)} - \epsilon_N\right) \tag{21}$$

with

$$F_{1N} = \sum_{i=1}^{N-1} a_{1i}^{(N)}\langle\Phi_1|H_1|\Phi_i\rangle \tag{22}$$

and

$$G_{1N} = H_{1N}\sum_{i=1}^{N-1} a_{1i}^{(N)}\langle\Phi_N|H_1|\Phi_i\rangle. \tag{23}$$

The coefficients $a^{(N-1)}$, $b^{(N-1)}$ and $c^{(N-1)}$ in equation (17) depend on $g^{(N)}$ through the presence of the coefficients $a_{1i}^{(N)}$, $i = 1, \ldots, N-1$. Since equation (17) is non-linear in $g^{(N-1)}$ and has two solutions, $g^{(N-1)}$ is chosen as the one closest to $g^{(N)}$ by continuity. In $\mathcal{H}^{(N-1)}$ the Hamiltonian $H^{(N-1)} = H_0 + g^{(N-1)}H_1$ is aimed to act in this subspace and used to describe the physical system therein.

The reduction process can be iterated step by step by projection from the space of dimension $N-1$ to $N-2$ and further, keeping at each step λ_1 equal to its initial value $\lambda_1^{(N)}$. One generates subsequently a succession of values of the strength parameter (coupling constant) $g^{(k)}$ at each iteration. At each step the projected wavefunction $\left|P\Psi_1^{(k)}\right\rangle$ is obtained from $\left|\Psi_1^{(k)}\right\rangle$ by elimination of a state $|\Phi_k\rangle$.

The evolution of the coupling constant g can be worked out in the continuum limit. For large N one goes over from $(k, k-1)$ to $(x, x-\mathrm{d}x)$. Writing equation (17) for two successive steps k to $k-1$ and $k-1$ to $k-2$, subtracting and going over to the continuum formulation x leads to the flow equation

$$\frac{\mathrm{d}g}{\mathrm{d}x} = -\frac{1}{2a(x)g(x)+b(x)}\left(\frac{\mathrm{d}c}{\mathrm{d}x} + \frac{\mathrm{d}b}{\mathrm{d}x}g(x) + \frac{\mathrm{d}a}{\mathrm{d}x}g(x)^2\right) \tag{24}$$

where $a(x)$, $b(x)$, $c(x)$ and $g(x)$ are the continuous extensions of the corresponding discrete quantities which depend on the dimension x of the space. Equation (24) is a non-linear differential equation which *a priori* can only be solved numerically.

In $\mathcal{H}^{(N-1)}$ the corresponding Hamiltonian $H^{(N-1)}$ is a local operator which is aimed to approximate non-local effects induced in H_{eff} through the existence of the second term in equation (6). This term induces a contribution to the renormalization of $g^{(N)}$ into $g^{(N-1)}$. It raises the question of the correspondence between energy-dependent H_{eff} and energy-independent Hamiltonians $H^{(k)}$ and shows that the procedure is approximate since these operators cannot be strictly equivalent. The correspondence between energy-dependent and energy-independent operators has been considered recently [27].

In the next section we apply the procedure to different models in order to analyse and discuss its practical efficiency.

4. Numerical applications

We aim to test the efficiency of the procedure which was described above by means of two examples. Starting from a full space with dimension N and a fixed eigenvalue λ_1 one reduces the space dimensions as described in section 3. Different values of N are considered as shown in tables 1 and 2.

4.1. Model 1

As a first application we consider a real symmetric tight-binding model which is generic for the description of many strongly correlated systems. Following the notations introduced above the Hamiltonian is degenerate and such that $H_0 = 0$, diagonal elements $\langle\Phi_i|H_1|\Phi_i\rangle = \beta$ and non-diagonal ones $\langle\Phi_i|H_1|\Phi_{i+1}\rangle = \langle\Phi_i|H_1|\Phi_{i-1}\rangle = \gamma$ which generate a coupling between nearest-neighbour states. By essence, a reduction of Hilbert space by means of perturbation expansions would be inefficient in this context, in particular when β and γ get large. Starting with an initial Hilbert space dimension N we apply the renormalization procedure described above to g starting from an initial value $g^{(N)}$. The evolution of the lowest eigenvalues and the flow of g are shown in table 1 for $g = 20$ and different values of N.

Table 1. Evolution of the coupling constant and the five lowest eigenvalues of the tight-binding matrix described in the text. Here $\beta = 1, \gamma = 0.5$. N is the initial space dimension, λ_1 is the ground state energy, λ_2–λ_5 are the energies of the lowest excited states.

N	n	g	λ_1	λ_2	λ_3	λ_4	λ_5
50	50	20	0.038	0.15	0.34	0.60	0.94
	20	3.73	0.04	0.165	0.37	0.65	0.99
	10	1.11	0.045	0.176	0.38	0.65	0.95
100	100	20	0.010	0.038	0.087	0.15	0.24
	50	5.3	0.010	0.04	0.09	0.16	0.25
	10	0.28	0.011	0.045	0.1	0.166	0.24
200	200	20	0.0024	0.01	0.022	0.04	0.06
	100	5.15	0.0025	0.01	0.022	0.04	0.06
	10	0.07	0.003	0.01	0.024	0.042	0.06

Table 2. Evolution of the coupling constant and the five lowest eigenvalues of the generalized tight-binding matrix described in the text. Here $\beta = 1, \gamma = 0.5, \delta = 0.5$, N is the initial space dimension, n is the dimension of the restricted spaces and g is the running coupling constant.

N	n	g	λ_1	λ_2	λ_3	λ_4	λ_5
50	50	20	−2.366	−2.360	−1.968	−1.942	−1.32
	30	21.8	−2.35	−2.30	−1.26	−1.08	0.53
	20	25.5	−2.27	−2.16	0.43	0.74	4.83
200	200	20	−2.491	−2.490	−2.464	−2.463	−2.42
	70	20.48	−2.49	−2.485	−2.27	−2.26	−1.91
	50	21.00	−2.485	−2.478	−2.067	−2.04	−1.38
	20	26.80	−2.39	−2.27	0.45	0.78	5.08
500	500	20	−2.50	−2.50	−2.49	−2.49	−2.49
	70	20.54	−2.495	−2.493	−2.28	−2.27	−1.92
	50	21.07	−2.49	−2.48	−2.07	−2.04	−1.38
	20	26.9	−2.39	−2.28	0.45	0.78	5.09

In order to quantify the quality of the spectrum we introduce the quantity

$$\Delta_i^{(N,n)} = \left|1 - \lambda_i^{(n)} \big/ \lambda_i^{(N)}\right| \tag{25}$$

where n stands for the size of the truncated space and i for increasing eigenenergies starting from the ground state $i = 1$.

For $N = 100$ and $n = 20$: $\Delta_i^{(100,20)} = 0.097, 0.092, 0.082, 0.068, 0.051$, $i = 1$–5 respectively.

For $N = 100$ and $n = 10$: $\Delta_i^{(100,10)} = 0.19, 0.16, 0.12, 0.07, 0.01$, $i = 1$–5 respectively.

The results show that the elimination of states $\{|\Phi_i\rangle\}$ which generate strong coupling matrix elements with the states in the remaining space has a strong influence on the renormalization of the Hamiltonian when the size of the reduced space gets small. For fixed n, $\Delta_i^{(N,n)}$ remains practically the same for any $N \geqslant 50$.

One also observes that the coupling parameter g decreases systematically and rather strongly with the dimensional decrease of space indicating that the coupling between the remaining states gets systematically weaker.

As expected further calculations show that better the stability of the spectrum of low-lying states the smaller the non-diagonal matrix elements.

4.2. Model 2

As a second example we consider a generalization of the preceding Hamiltonian which is generated by adding a coupling between next-nearest-neighbour states $\langle\Phi_i|H_1|\Phi_{i+2}\rangle = \langle\Phi_i|H_1|\Phi_{i-2}\rangle = \delta$. One expects that the further coupling to next-nearest states increases the correlation between states. The outcome of the diagonalization is shown in table 2 for different values of the initial space dimensions N. The stability of the lowest eigenvalues during the renormalisation procedure is preserved up to some minimal dimension $n_{\min}$. The ration $n_{\min}/N$ decreases with increasing N.

Using $\Delta_i^{(N,n)}$ as in the previous example one finds the same general trends. For $N = 500$ down to $N = 50$ and fixed $n = 20$, $\Delta_1^{(N,n)} \sim 0.04$ and $\Delta_2^{(N,n)} \sim 0.09$.

In the cases $N = 200$ and $N = 500$ the eigenvalues $\lambda_3, \lambda_4, \lambda_5$ change and increase for $n \leqslant n_{\min}$ with $70 < n_{\min} < 100$. Due to the strong coupling of the degenerate eigenstates of $H_0 = 0$, the renormalization of g cannot counteract the effect induced by the elimination of states. The comparison of the behaviour of the eigenenergies between model 1 and model 2 show that the deviations are stronger the larger the number of non-diagonal matrix elements. As a consequence the truncation process should be stopped at $n_{\min}$ when quantitative effects are sizable. One may note that $n_{\min}$ gets independent of N for some $N_{\min}$ as it is the case in model 1.

The present numerical investigations concern systems in which the states $\{|\Phi_i\rangle, i = 1, \ldots, N\}$ are degenerate and strongly coupled to each other, either directly or indirectly. The strong coupling is also seen through the fact that the ground state energy does not completely stabilize when the initial dimension of the Hilbert space N increases from 50 to 500 as seen in tables 1 and 2.

In practice and generally speaking a meaningful truncation algorithm may not necessarily ground on the systematic elimination of those states whose diagonal matrix elements lie highest in energy. The physical low-lying states are the states of interest. An importance sampling sorting out those states which have the strongest components on the physical low-lying states due to strong non-diagonal matrix elements should be kept in the final space of states.

5. Exceptional points and fixed points

We now show that in the present scheme, exceptional points which correspond to values of g for which perturbation expansions diverge [16, 21] correspond to fixed points in the renormalization process. The result is valid for any type of Hamiltonian which depends on a coupling constant g.

It has been rigorously established that the eigenvalues $\lambda_k(g)$ of an Hamiltonian $H(g) = H_0 + gH_1$ are analytic functions of g with only algebraic singularities [16, 21, 22]. They get singular at the so-called exceptional points $g = g_e$ which are first-order branch points in the complex g-plane. Branch points appear if two (or more) eigenvalues get degenerate. This can happen if g takes values such that $H_{kk} = H_{ll}$ where $H_{kk} = \langle\Phi_k|H|\Phi_k\rangle$, at a so-called level crossing. As a consequence, if a level belonging to the $P\mathcal{H}$ subspace defined above crosses a level lying in the complementary $Q\mathcal{H}$ subspace, the perturbation development constructed from $H_{\text{eff}}(E)$ diverges [16]. Exceptional points are defined as the solutions of [22]

$$f(\lambda(g_e)) = \det[H(g_e) - \lambda(g_e)I] = 0 \tag{26}$$

and

$$\frac{\mathrm{d}f(\lambda(g_e))}{\mathrm{d}\lambda}|_{\lambda=\lambda(g_e)} = 0 \tag{27}$$

where $f(\lambda(g))$ is the secular determinant. It is now possible to show that exceptional points can be identified as fixed points in the truncation procedure. If $\{\lambda_i(g)\}$ are the set of eigenvalues the secular equation can be written as

$$\prod_{i=1}^{N}(\lambda - \lambda_i(g)) = 0. \tag{28}$$

Consider $\lambda = \lambda_p(g)$ which satisfies equations (26) and (28). Then equation (27) can only be satisfied if there exists another eigenvalue $\lambda_q(g)$ such that $\lambda_p(g) = \lambda_q(g)$, hence if a degeneracy appears in the spectrum. This is the case at an exceptional point.

Going back to the algorithm described above consider the case where some eigenvalue λ_c, not necessarily the energy of the ground state, is either constrained to be fixed as above or independent of g for any value of g and gets degenerate with some other eigenvalue $\lambda_i^{(k)}(g = g_e)$ at some step k in the space reduction process. Since λ_c is constrained to be constant

$$\lambda_i^{(k)}(g_e) = \lambda_i^{(l)}(g'_e) \tag{29}$$

which is realized in any projected subspace of size k and l containing states $|\Psi_c\rangle$ and $|\Psi_i\rangle$, the eigenvectors corresponding to λ_c and λ_i. In the continuum limit for large values of N and considering the subspaces of dimension x and $x + \mathrm{d}x$ one can write

$$\frac{\mathrm{d}\lambda_c}{\mathrm{d}x} = 0 = \frac{\mathrm{d}\lambda_i(x)}{\mathrm{d}x}. \tag{30}$$

Consequently

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}g_e}\frac{\mathrm{d}g_e}{\mathrm{d}x} = 0. \tag{31}$$

Due to the Wigner–Neumann avoided crossing rule the degeneracy of eigenvalues is generally not fulfilled for real values of the coupling constant and the derivative of λ_i with respect to g vanishes. There exist however specific situations, systems with symmetry properties [16, 25] or infinitely large ones [23] for which degeneracy for real g can occur. In these cases equation (31) is realized if

$$\frac{\mathrm{d}g_e}{\mathrm{d}x} = 0 \quad \text{and} \quad \frac{\mathrm{d}\lambda_i}{\mathrm{d}g_e} \neq 0. \tag{32}$$

The second relation works if crossing takes place and g_e is a fixed point in the sense of renormalization theory as shown by the first relation.

Equation (32) establishes the connection between exceptional and fixed points in the framework of the present approach. State degeneracy due to level crossing is indeed a signature for the existence of phase transitions [24], perturbation expansions break down at these points. The ground state wavefunction changes its properties when the (real) coupling constant g crosses the exceptional point g_e. There the eigenstates exchange the main components of their projection on the set of basis states $\{|\Phi_i\rangle, i = 1, \ldots, N\}$. It is worthwhile to emphasize that this result is not restricted to the ground state, it is valid at any physical level crossing in which one of the eigenvalues stays constant for any value of the coupling constant. We have tested this property on several systems. A simple example is presented below.

6. Example

We consider a quantum spin 1/2 ladder with four sites $(1, 2, 1', 2')$. The sites $(1, 2)$ and $(1', 2')$ are located on parallel lines, on a ladder with two rungs $(1, 1')$, $(2, 2')$. Their antiferromagnetic interaction is described by the Hamiltonian [26]

$$H = 2J_1(S_1S'_1 + S_2S'_2 + \alpha_2(S_1S_2 + S'_1S'_2) + \alpha_3S'_1S_2 + \alpha_4S_1S'_2) \tag{33}$$

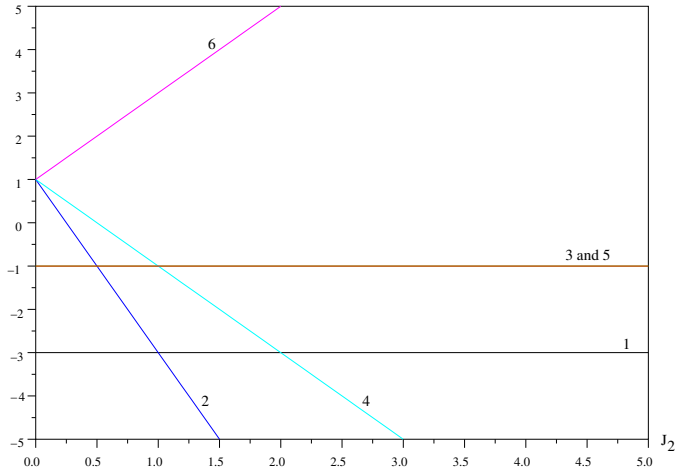

Figure 1. Evolution of the six eigenenergies of the Hamiltonian equation (33) with the strength parameter $J_2 = \alpha_2 J_1$, $J_1 = 1$. The numbers in the figure label the different states. See discussion in the text.

Table 3. Behaviour of the three lowest eigenenergies corresponding to the model described by the quantum spin Hamiltonian given by equation (33) for $\alpha_2 = \alpha_3 = \alpha_4 = 0.5$ and $g = 2J_1 = 20$. See discussion in the text.

N	n	g	λ_1	λ_2	λ_3
6	6	20	−30	−10	−10
	5	20	−30	−10	−10
	4	20	−30	−10	−10
	3	20	−24.14	−10	4.14

where J_1 (here $2J_1 = g$ in the former notation) is a positive coupling constant which will be renormalized through the Hilbert space reduction procedure. The quantities α_2, α_3 and α_4 are positive quantities. In the present case $H_0 = 0$ and the basis vectors are chosen as $|m_1, m_1', m_2, m_2'\rangle$ where $m_i = +(1/2)$ or $-(1/2)$ is the projection of the spin $1/2$ on the quantization axis at site i. The subspace corresponding to $M_{\rm tot} = 0$ where $M_{\rm tot}$ is the sum of the spin projections contains six states. The diagonalization for fixed J_1 shows that the eigenstates cross each other at specific values of α_2 as shown in figure 1.

As an illustration of the fixed point property discussed above we consider the crossing point between the state labelled 2 and the degenerate states labelled 3 and 5 in figure 1. This is an exceptional point. The numerical result shown in table 3 confirms that it is also a fixed point of the renormalization procedure induced by space truncation. As expected, $g = 2J_1$ does not change when the size of the Hilbert space is reduced from $n = 6$ to $n = 3$.

The three lowest eigenenergies are conserved up to $n = 4$. For $n = 3$ the ground state and the second excited state energies are strongly affected. As already discussed above at the end of section 4, this is due to the fact that the fourth basis vector is strongly coupled to the first one, both being large components of the ground state wavefunction.

7. Conclusions

In summary, we developed a non-perturbative effective formalism of the bound state many-body quantum problem based on a reduction process of the dimensions of the initial Hilbert

space by means of a truncation procedure. This leads to the generation of effective Hamiltonians which act in the reduced spaces. They are obtained through a step-by-step reduction procedure and are characterized by renormalized coupling parameters. The parameters are obtained by means of the implementation of constraints on the physical spectrum in the successive truncated spaces, here the ground state energy of the system. We presented and analysed the virtues and limitations of the approach on two examples describing strongly coupled and highly degenerate systems which albeit simple are characteristic for strongly correlated systems and for which perturbative expansions are meaningless. We further showed the relationship which exists between exceptional points corresponding to level crossings in the spectrum where perturbation expansions break down, and fixed points of the coupling constants. These points characterize the so-called quantum phase transitions at zero temperature. The formalism can be applied to other physical quantum systems such as atoms, molecules, aggregates as well, whatever the explicit form of the Hamiltonian. It can be extended to systems characterized by several coupling constants and systems at finite temperature. Effective operators acting in reduced space can be worked out. We shall present these developments in the forthcoming work.

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